CVA: Default Probability ain't matter?

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Abstract

CVA can be priced using market implied 'risk-neutral' or historical 'real-world' parameters. There is no consensus in the market as to which approach to use. This paper illustrates in simple terms the fundamental differences between the two approaches and the consequences of using each of them.

Introduction

Most readers will know that CVA is an adjustment made to the price (MTM) of a transaction to account for the credit risk embedded in it.

$$\mathrm{MTM} = \mathrm{MTM}_{RiskFree} - \mathrm{CVA}. \tag{1}$$

That adjustment can be done one-way, accounting for the credit risk of one counterparty only, or two-way, netting off the credit risk of both counterparties[1]. In this text, we are going to explore one-way CVA.

Under certain assumptions explained in Appendix **??**, a common mathematical expression for CVA is:

$$\mathrm{CVA} = (1 - R) \int_0^T E_t P_t B_t \mathrm{d}t \tag{2}$$

where R is the recovery rate, E_t is the expected exposure in the portfolio at time t, P_t is the expected default probability at time t, B_t is the discount factor for a cash flow at time t and T is the the portfolio maturity.

As reported in "Reflecting the Fair Value of Financial Instruments, a Survey" by Ernst & Young, there is discrepancy as to what should be used for the default probability P_t and the recovery rate R. Some professionals believe that we should use the default probability

[1] "CVA Demystified", Ignacio Ruiz, www.iruizconsulting.com

that can be implied from the CDS market, the so-called ‘risk-neutral’ approach, whilst other professionals suggest that we should use the default parameters given by internal models, typically based in historical data, the so called ‘real-world’ approach.

Which of these methods is correct, and which is wrong? Why? What are the implications of using each of them?

This paper gives answer to those questions.

Difference between both approaches

In principle, both approaches can be seen as correct *as long as their meaning is well understood.*

On the one hand, under the risk-neutral approach, some basic assumptions lead to the following expression for CVA (see Appendix **??**):

$$\mathrm{CVA}_{RiskNeutrual} \approx \int_0^T E_t \cdot s \cdot \mathrm{d}t, \tag{3}$$

where s is the credit spread of the counterparty at stake. This formula clearly shows that under the ‘risk-neutral’ approach, CVA is the actual *cost of hedging* the expected exposure of the portfolio. We can say that, subject to the calculation of E_t being accurate, if we hedge all portfolios via CDSs with a notional profile described by E_t, then, on average, the income we would receive from the hedges in the event of default will compensate for the credit losses.[2]

On the other hand, under the ‘real-world’ approach, we will plug into Equation **??** the recovery rate and the default probability given by our models (e.g., a rating migration model), and calculate the CVA accordingly.

The reader can see how in the ‘risk-neutral’ approach all that it is needed is a good model for the portfolio expected exposure; the rest is dictated by the market. However, in the ‘real-world’ approach, we need a model not only for the expected exposure, but also for counterparty default and recovery rate. If the recovery rate and the default probability are different to the market implied, then both CVA numbers will be different in general.

To hedge or not to hedge

In any case, however CVA is calculated, the following happens afterwards in a typical banking set up:

[2] Many readers know that, in practice, things are not as simple as this due to market inefficiencies, lack of liquidity and many other issues. However, as an approximation, we can say that this statement is true.

1. The transaction trading desk will be charged according to this CVA number.
2. Depending on the bank policy, there are two basic options. The bank can hedge out the credit risk via the CDS market, or it can accrue the CVA cash to compensate for expected future losses.

The following sections will explain in simple terms how each 'risk-neutral' and 'real-world' approach are strongly linked to each of the two options.

Boxing managers also face "Risk Neutral" V. "Real"

In order to further illustrate things we could use a simple analogy that should make the reader familiar with this matter.

Let's say, for example, that we are into the business of promoting boxers. We have a boxer in our book, called Max, who is fighting soon with another boxer. It so happens that both boxers have exactly the same track record of victories, similar techniques, equivalent level, same weight, same... everything. Based on this, historical analysis tells us that chances of victory are split at 50/50 chance.[3]

However, our boxer Max unfortunately twisted his wrist recently, so it is not clear if he will be able to be in top shape for the fight. This information is public and, as a result, sport betting houses are giving him only a 33% chance of winning the fight.

On top of this, being his promoter, we know very well the state of his injury. We know that he is recovering very fast, and not only that, he is now finishing a new secret training program that has increased his fitness level to a new spectacular level. With all this in mind, we estimate that his chance of winning the fight is 66%.

Summarising, we have the following sets of analysis for the fight:

- The market gives a 33% success probability
- Our internal model gives a 66% success probability

On the financial side we, as promoters, have invested around $1 million in Max. If he wins the fight we are going to make zillions out of publicity, etc, etc, but if he loses, we expect to lose the whole $1 million. There are two things that we can do:

1. We can use the sport betting market to hedge out our exposure to the event.
2. We can leverage from our internal knowledge (i.e., his recovery from the injury plus his 'secret' training program) and try to make money out of it.

[3] For simplicity, let's say that the outcome of the fight is two folded: one of the two contenders must win.

Regarding point 1, in order to hedge out our exposure, all we have to do is to place a \$0.33 million bet against Max.[4] In this way, if Max wins, we will make zillions minus \$0.33 million and if Max looses, we will recover the \$1 million. As a result, we will be risk hedged at a price of \$0.33 million.

Regarding point 2, in order to leverage from our internal models, we need bet further more for Max's victory, as we think that the 33% chance given by the market is too low. However, if our internal models say that the chance of Max victory are, say, only 10%, then we will have to bet against Max's victory. In this case, we may still make a loss if the actual event outcome is against our bet, but the idea is that if we believe that our internal model is better than that of the market, and we play this game several times, we will make money in average.

The main point to highlight here is the following: when we want to hedge out our risk as boxing promoters, *the actual probability of Max's victory is irrelevant*. The only thing that matters is the price of hedging out the risk, because even if we think that the market is pricing risk incorrectly, to hedge our exposure we still need to pay the market price: 33% of notional.

As a natural extension to that, if we try to hedge by betting a number different to \$0.33 million, we will not be completely hedged.

That is, regarding risk hedging, my view on event probability does not matter, what matters is the market'view; that is *the market price of risk*.

CVA: Default Probability ain't matter?

As the reader can already see, this boxing story and CVA share some similarities. In CVA, typically there are also two sources of information to estimate default parameters and to price credit risk: market implied probabilities via CDS prices and internal models. Which one of these should an institution use to price CVA?

The answer depends on what the institution wants to achieve. If it wants to hedge out the credit risk, market-implied information must be used, regardless of its view on whether the market price of credit risk is correct. In this case, *default probability ain't matter*. However, if it wants to accrue cash to account for expected future credit losses, then it should use its own internal models, whatever they are... and hope that those models are correct!

[4] A 33% chance in a bet with only two possible outcomes gives a 2/1 betting odd, assuming the betting market is perfectly efficient, the bookies margins are zero, etc, for simplicity.

A Appendix: CVA mathematical formulation

The mathematical expression for CVA at time t_0 is:

$$\mathrm{CVA}_{t_0} = \mathbb{E}\left[\int_{t_0}^{t_0+T} E_t P_t B_{t_0,t}(1-R_t)\mathrm{d}t\right] \tag{4}$$

where $\mathbb{E}(\bullet)$ is the expectation operator, E_t is the exposure of the portfolio at hand at time t, P_t is the default probability of the counterparty at hand at time t, $B_{t_0,t}$ is the discount factor at time t_0 for a cash flow at t, R_t is the portfolio recovery rate at time t. If $\Psi_t(E,P,R))$ is the joint probability function of all those variables at time t, then

$$\mathrm{CVA}_{t_0} = \int_{t_0}^{t_0+T} E_t P_t B_{t_0,t}(1-R_t)\Psi_t(E,P,R)\mathrm{d}t \tag{5}$$

Finally, if

$$\Psi_t(E,P,R)) = \Psi_t^E(E)\Psi_t^{PD}(P)\Psi_t^R(R) \tag{6}$$

that is, if all variables E, P, and R are independent, then the CVA value can be expressed as

$$\mathrm{CVA}_{t_0} = \int_{t_0}^{t_0+T} \overline{E}_t \overline{P}_t B_{t_0,t}(1-\overline{R_t})\mathrm{d}t \tag{7}$$

where $\overline{X} \equiv \mathbb{E}(X)$. It is also common habit to assume that the recovery rate is constant over time, in which case we get to the final well known expression for the CVA:

$$\mathrm{CVA}_{t_0} = (1-R)\int_{t_0}^{t_0+T} \overline{E}_t \overline{P}_t B_{t_0,t}\mathrm{d}t \tag{8}$$

B Appendix: CVA reduction to CDS spread

Let's price a CDS (per unit of notional) with a few approximations that are usually acceptable.

Let's say that we have a CDS in a world of continuum payments of coupons s with coupon payments in a infinitesimal time step $\mathrm{d}t$ of $s \cdot \mathrm{d}t$. Those coupons will be paid as long as the CDS reference entity has not defaulted before t. This event has a probability S_t. Finally, let's say that the risk-free discount factor for a cash flow at t is B_t. The present value of a infinitesimal coupon payment at t is, then, $s \cdot S_t \cdot B_t \cdot \mathrm{d}t$.

In the event of default at $\mathrm{d}t$, the protection buyer will receive a payment $1 - R_t$. This event has a probability $P_t \cdot \mathrm{d}t$, where

$$P_t \cdot \mathrm{d}t = -\frac{\partial S_t}{\partial t}\mathrm{d}t \tag{9}$$

The present value of this payment is $(1 - R_t)P_tB_t$.

The price at time 0 (i.e., the spread s) of the CDS that matures at time T is then given by equating both the premium and contingent payment legs:

$$\int_0^T [sS_tB_t - (1 - R_t)P_tB_t]\,\mathrm{d}t = 0 \tag{10}$$

If we say that we buy a number of CDSs, so that the notional is time dependent and denoted by E_t, then Equation **??** and Equation **??** lead to the following expression for CVA:

$$CVA = \int_0^T E_tsS_tB_t\mathrm{d}t \tag{11}$$

In many cases, interests rates are very low so that $B_t \approx 1$, and often $S_t \approx 1$ too. Then,

$$CVA \approx \int_0^T E_ts\mathrm{d}t \tag{12}$$